Announcements

- Section next week prelim review: it is optional, and feel free to attend the section of your choice

- HW and section question solutions prelim review solutions all on canvas
- Prelim 2: Tuesday March 24^{th} in Statler auditorium, email from Amy if you have a conflict
  - Topics: stable matching, flows and applications and NP-completeness
  - Information sheet on topics and sample question & solutions in canvas

- HW8 will be divide and conquer, due Friday April 10

Lessons from hw7

in NP include 1. hint needed so you can verify
2. easy to check in poly time

Wednesday Master Theorem for solving divide and conquer recurrences

time $\quad T(n) = q\,T(n/2) + c \cdot n^{\alpha}$ $\quad$ (α, c, q constants)

n = input size

① $q > 2^{\alpha}$ $\quad$ time $T(n) = O(n^{\log_2 q})$ $\quad$ integer multiply, matrix multiply

* ② $q = 2^{\alpha}$ $\quad$ time $O(n^{\alpha} \log_2 n)$ $\quad$ merge sort

* ③ $q < 2^{\alpha}$ $\quad$ time $O(n^{\alpha})$

Example 0: binary search on sorted array

$[a_1 \ldots a_n]$ sorted & input x $\quad$ is $x = a_i$ some i

$a_{n/2} \geq x$ $\quad$ $T(n) = T(n/2) + O(1)$ $\quad$ $1 = 2^0$ $\quad$ option ②

Randomized Median finding (and Quicksort)

Input $a_1 \ldots a_n$ find k^{th} smallest (assume all different)

Option 1: sort & take k^{th} item $O(n \log n)$

Idea: pick a_i at random

find $S_- = \{j : a_j < a_i\}$ $S_+ = \{j : a_j > a_i\}$

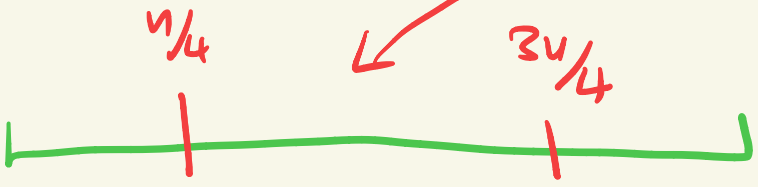
1. if $|S_-| + 1 = k \Rightarrow$ return a_i
2. if $|S_-| + 1 > k \Rightarrow$ find k^{th} item (S_-)
3. if $|S_-| + 1 < k \Rightarrow$ find $(k - |S_-| - 1)^{th}$ item in (S_+)

Suppose each iteration

$$|S_-| \;\&\; |S_+| \le \tfrac{3}{4}n$$

i.e. a_i in middle half

a_i picked here



sorted version of a_is

then $$T(n) \le T\left(\frac{3n}{4}\right) + \underline{O(n)}$$

comparisons $$C(n) \le C\left(\frac{3n}{4}\right) + \underline{n-1}$$

to get S_- & S_+

Join by Web PollEv.com/evatardos772

Which is the tightest valid upper bound on the number of comparisons in median finding, assuming each iteration is "lucky"

A. $n-1$

B. $4n$

C. $8n$

D. $O(n \log n)$

E. $O(n^2)$

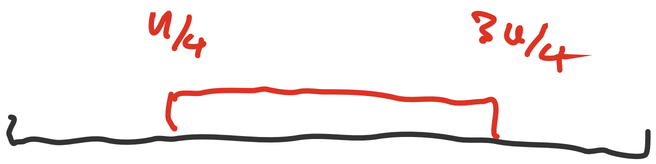
comparison recurrence

$$C(n) \le C\left(\frac{3n}{4}\right) + (n-1)$$

$$= n-1 + \frac{3n}{4} - 1 + \frac{3^2 n}{4^2} - 1 \le n + \frac{3n}{4} + \left(\frac{3}{4}\right)^2 n + \left(\frac{3}{4}\right)^3 n + \dots$$

$$\le n \cdot \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i = n \cdot \frac{1}{1-3/4} = 4n$$

Analyzing median: phases



sorted version

Pr(a_i picked in middle half) = $\frac{1}{2}$

Option 1: if $|S_-|$ or $|S_+| < n/4$ try again

X = # tries till a_i in middle half

$$E(X) = 1 \cdot Pr(X=1) + 2\,Pr(X=2) + \dots$$
$$= Pr(X \geq 1) + Pr(X \geq 2) + Pr(X \geq 3) + \dots$$
$$= 1 + \frac{1}{2} + \frac{1}{4} + \dots = 2$$

Pr(event) = p
tries till it happens $1/p$

Algorithm: pick a_i random repeatedly till $|S_-|$ & $|S_+| \geq n/4$
& then recurse on one side

$C(n)$ = # comparisons expected till we can recurse $\quad 2(n-1)$

Expected running time of Median Finding

Exected # comparisons $\underline{C(n) = C(n\tfrac{3}{4}) + 2(n-1)}$

$\Rightarrow \quad C(n) \leq 8n$

Option 2: OK to recurse also when not in middle half

Running time $O(n)$

Summary of new Master theorem

$T(n) = q\,T(\frac{n}{\beta}) + c \cdot n^{\alpha}$

levels $\log_\beta n$

important β constant

1. $q > \beta^\alpha \quad \Rightarrow \quad T(n) = O(n^{\log_\beta q})$
2. $q = \beta^\alpha \quad \Rightarrow \quad T(n) = O(n^\alpha \log_\beta n)$
3. $q < \beta^\alpha \quad \Rightarrow \quad T(n) = O(n^\alpha)$

today $q = 1$ $\beta = 4/3$ & $\alpha = 1$ indeed $(4/3)^1 > 1$